Instability Induced Renormalization

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Abstract

It is pointed out that models with condensates have nontrivial renormalization group flow on the tree level. The infinitesimal form of the tree level renormalization group equation is obtained and solved numerically for the ϕ^4 model in the symmetry broken phase. We find an attractive infrared fixed point that eliminates the metastable region and reproduces the Maxwell construction.

We have two systematic nonperturbative methods to handle multi-particle or quantum systems, the saddle point approximation and the renormalization group. Our goal in this letter is to combine these two apparently independent approximation methods in order to obtain a better understanding of the instabilities and first order phase transitions.

We start with the path integral,

$$Z=\int[\mathcal{D}\phi]\,e^{-\frac{1}{\hbar}S[\phi]}, \tag{1}$$

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over the configurations $\phi(x)$. We assume the presence of UV and IR cutoffs thus the dimension of the domain of integration is large but finite.

The saddle point approximation to (1) coincides with its perturbative expansion when the saddle points are trivial, $\psi = 0$. In this case the elementary excitations, the quasiparticles, are characterized by the eigenfunctions $\phi_n(x)$ of the inverse propagator, $G^{-1}(x,y) = \delta^2 S[\phi]/\delta\phi(x)\delta\phi(y)_{\big|\phi=\psi}$. The perturbation expansion is applicable for fixed values of the cutoffs if the restoring force of the fluctuations to $\phi(x) = \psi(x) = 0$ is nonvanishing, i.e. the eigenvalues of the inverse propagator, λ_n, are positive. When the absolute minimum of the action is reached at $\phi(x) = \psi(x) \neq 0$ then we follow the strategy of the saddle point expansion and the elementary excitations are the fluctuations around $\psi(x) \neq 0$. The saddle point expansion is applicable as long as $\lambda_n > 0$. The zero modes, the elementary excitations with $\lambda_n = 0$, correspond to continuous symmetries and are integrated over exactly. If the inverse propagator has too many small eigenvalues the saddle point expansion breaks down and strong fluctuations develop around ψ as we remove one of the cutoffs. This possibility brings us to the renormalization group method which is supposed to deal with such problems [1].

The basic idea of the renormalization group is the subsequent integration in (1),

$$Z = \int d\phi_1 \int d\phi_2 \cdots \int d\phi_N e^{-\frac{1}{\hbar}S_N[\phi]}, \tag{2}$$

where the field is expanded according to a suitable chosen basis, $\phi(x) = \sum_{n=1}^{N} \phi_n \Phi_n(x)$, and $N < \infty$ plays the role of the UV cutoff. The effective theory for the first N' modes is defined through the effective action $S_{N'}$ by the help of the blocking transformation for $\Delta N = N - N'$ number of modes,

$$e^{-\frac{1}{\hbar}S_{N'}[\phi]} = \int d\phi_{N'+1} \cdots \int d\phi_N e^{-\frac{1}{\hbar}S_N[\phi]}. \tag{3}$$

The elimination step is usually followed by a rescaling in order to restore the cutoff to its original value. This rescaling, together with its result, the anomalous dimension, is an important device to distinguish the trivial (tree-level) and the dynamical (loop-level) scale dependence.

The fluctuations of the IR modes with $n < N$ can be described within the effective theory given by the action

$$S_N[\phi] = \sum_j g_j(N) s_j[\phi, \partial_\mu\phi, \Box\phi, ...], \tag{4}$$

where s_j is a suitable chosen complete set of local functionals without keeping the UV modes $n > N$ present explicitly. Thus the IR modes decouple from the UV ones since the correlations generated by the latters are contained in the numerical value of the effective coupling constant $g_j(N)$. This trivial observation gives rise a powerful approximation scheme by the truncation of (4) when *any* IR field configuration can be used to reconstruct the *same* overdetermined effective action. In the infinitesimal form of the renormalization group method, [2], where $\Delta N << N$ the small parameter of the loop expansion in (3) is $\hbar\Delta N/N$, thus we expect that the saddle point approximation is applicable in (3). The Wegner-Houghton equation [2] is obtained by eliminating the plane waves within the shell $k - \Delta k < |p| < k$ of the momentum space,

$$
\frac{\partial S_k[\phi]}{\partial k} = \frac{\hbar}{\Delta k}\left[\frac{1}{2}\mathrm{Tr}_{k,\Delta k}\frac{\delta^2 S[\phi]}{\delta\phi\delta\phi} - \frac{1}{\hbar}\frac{\delta S[\phi]}{\delta\phi}\cdot\left(\frac{\delta^2 S[\phi]}{\delta\phi\delta\phi}\right)^{-1}\cdot\frac{\delta S[\phi]}{\delta\phi}\right]
$$

$$
-\frac{\eta+d}{2}\partial_\mu\phi\cdot\frac{\delta S[\phi]}{\delta\partial_\mu\phi} + \frac{2-\eta-d}{2}\phi\cdot\frac{\delta S[\phi]}{\delta\phi}, \tag{5}
$$

where the trace and the "$\cdot$" operation is taken in the subspace of the eliminable modes. The second line generates the rescaling assuming the anomalous dimension η (input) in dimension d. The second order approximation is used for the action as the functional of the eliminable field variables which is applicable only if the saddle point amplitude is $O(\hbar)$. Our interest in this work is the case where there is a saddle point during the elimination of the modes and we shall see that there is no gurantee in general that the saddle point remains $O(\hbar)$ during the evolution. Thus we need a more reliable evolution equation which is applicable without imposing the condition $O(\hbar)$ on the saddle point amplitude.

Systems with instabilities display fluctuations with large amplitudes which can be taken into account by means of the saddle point approximation. In order to demonstrate the importance of the saddle point during the blocking on the renormalized trajectory we shall consider the ϕ^4 model in the symmetry broken phase in a box with linear size L and with periodic boundary conditions. The bare action of this model is $S_\Lambda = \int d^dx[\frac{1}{2}(\partial_\mu\phi)^2 + U_\Lambda(\phi)]$ where $U_\Lambda(\phi) = -m^2/2\phi^2 + g\phi^4/4!$ (Λ is the UV cutoff) and we search for the vacuum in the presence of the constraint $L^{-d}\int d^dx < \phi(x) >= \Phi$. The constrained vacuum displays spinodal instability or metastability when it is unstable against fluctuations with infinitesimal or finite amplitude, respectively. The spinodal instability can be detected by local methods, mode by mode inspection. In fact, each fluctuation of the form

$$
\psi_k(x) = \tilde\psi_k e^{ik_\mu x_\mu} + \tilde\psi^\star_k e^{-ik_\mu x_\mu} = 2\rho_k\cos(k_\mu x_\mu + \alpha_k) \tag{6}
$$

with infinitesimal amplitude, $\rho_k \approx 0$, represents an independent unstable mode of the tree level theory when $p^2 < m^2 - g\Phi^2/2$. The Legendre transform of the tree level theory suggests that the vacuua with $2m^2/g < \Phi^2 < 6m^2/g$ are metastable. The vacuum is stable when $\Phi^2 > 6m^2/g$. It is difficult to identify the possible instabilities of the true vacuum where the local, mode by mode analysis is unreliable. This is because the decay of the unstable vacuum udergoes large amplitude modifications either in the metastable or the spinodal instable region.

We show now that the renormalization group can be used to deal with the modes of the unstable vacuum in a simple one by one manner. The blocking transformation we employ consists of the elimination of the modes with momentum $k - \Delta k < |p| < k$, where k is the current cutoff and Δk is a momentum parameter smaller than any characteristic momentum scale of the system. The effective coupling constants can be defined in the leading order of the gradient expansion by $U_k(\phi) = \sum_j g_j(k)/j!\phi^j$. Since any infrared field configuration should yield the same effective action we take the simplest choice, a homogeneous field $\phi(x) = \Phi$.

The loop expansion for (3) yields the general result $S_{N'} = \sum_j \hbar^j S^{(j)}_{N'}$. The perturbation expansion retains the term $O(\hbar^j)$ with $j > 0$ and the tree level, $O(\hbar^0)$ contribution represents a non-perturbative piece what must be considered *before* the perturbative pieces are taken into account. Since the tree level expressions are different from the loop corrections the tree

level renormalization, if exists, consists of essentially new and different expressions than the loop contributions which have been considered so far [4]. Our constraint generates non-trivial saddle points and we consider here the leading order, tree level renormalization only. In this approximation the naive scaling is correct ($\eta = 0$, $Z = 1$), and we ignore the rescaling step of the renormalization group method for the sake of simplicity.

The saddle point of the blocking transformation can be written in the form $\psi_k(x) = \sum' \tilde{\psi}_p e^{ipx}$ where the prime denotes the summation for $k - \Delta k < |p| < k$. The blocked potential is given by

$$L^d U_{k-\Delta k}[\Phi] = -\hbar \ln \left[\int [\mathcal{D}\eta] e^{-\frac{1}{\hbar} S_k[\Phi+\eta]} \right] = \min_{\{\psi\}} \left[\int d^d x \left(\frac{1}{2}(\partial_\mu \psi)^2 + U_k(\Phi + \psi) \right) \right] + O(\hbar) \quad (7)$$

where the fluctuation η contains only the modes within the shell $[k - \Delta k, k]$. It is worthwhile noting that (7) reduces to the usual local potential approximation of the Wegner-Houghton equation [5] if the saddle point vanishes. There might be several minima ψ in which case one should sum over them. We retain only the *single plane wave* saddle points, (6). The motivation of this rather drastic approximation is the assumption that the saddle point (6) seems to be optimal from the point of view of the energy-entropy ballance. To see this first note that when $\Delta k \to 0$ the saddle point is non-vanishing in the momentum space on the sphere with radius k only. As far as the saddle point on this sphere is concerned, the introduction of other additional plane waves would increase the kinetic energy. The phase $\alpha_k = \alpha_{-k}$ is a zero mode for each plane waves, corresponding to the translation invariance. The more involved saddle points built by several plane waves have the same single translational zero mode. Thus the kinetic energy-entropy balance is optimized for the plane waves. The resulting tree-level blocking relation is

$$\begin{aligned} U_{k-\Delta k}(\Phi) &= \min_{\{\rho\}} \left[k^2 \rho^2 + \frac{1}{2} \int_{-1}^{1} du \; U_k(\Phi + 2\rho \cos(\pi u)) \right] \\ &= k^2 \rho_k^2 + \frac{1}{2} \int_{-1}^{1} du \; U_k(\Phi + 2\rho_k \cos(\pi u)). \end{aligned} \quad (8)$$

We followed the tree level evolution of the potential by performing numerically the minimization in (8) at each blocking step. The most interesting aspect of the result is that, starting from $k = \sqrt{m^2} - \Delta k$, the saddle points ψ_k are nonvanishing (and consequently the potential receives a nontrivial renormalization) for $0 < \Phi^2 < \Phi_{vac}^2(k) = 6(m^2 - k^2)/g$. Thus the mode coupling provided by our successive elimination method extends the spinodal instability over the vacuua which are seen as metastable by the tree level Legendre transformation. This can happen because once the tree level contributions are found at modes with $p^2 < m^2$ for certain values of Φ^2 the saddle point contributions renormalize the potential in such a manner that spinodal instability occures for other, larger values of Φ^2.

Other important results of the numerical analysis are in order:
(i) the saddle point amplitude is a linear function of the field, $2\rho_k = -\Phi + \Phi_{vac}(k)$;
(ii) we find the potential

$$U_k(\Phi) = -\frac{1}{2} k^2 \Phi^2 - \frac{3}{2g}(m^2 - k^2)^2 \quad (9)$$

whenever the saddle points are nontrivial, i.e. in the unstable region, see Fig.1, where the evolution of the potential with $m^2 = 0.1$ and $g = 0.2$ is shown;

(iii) the blocking converges as $\Delta k \to 0$ despite the apparent absence of Δk in (8);
(iv) we checked that the previous results are *universal* with respect to the choice of the symetry breaking bare potential.

It is possible to understand the result (iii) if we write (8) for two subsequent steps and take the difference:

$$U_{k-2\Delta k}(\Phi) = U_{k-\Delta k}(\Phi) - \Delta k \left[2k\rho_k^2 + 2k^2\rho_k\partial_k\rho_k + \frac{1}{2}\int_{-1}^{1} du\ \partial_k U_k\left(\Phi + 2\rho_k\cos(\pi u)\right) \right.$$
$$\left. + \int_{-1}^{1} du\ \cos(\pi u)\ \partial_k\rho_k\ \partial_\phi U_k\left(\Phi + 2\rho_k\cos(\pi u)\right)\right] + \mathcal{O}(\Delta k)^2, \qquad (10)$$

which shows explicitely that the correction to the potential due to one blocking step is proportional to Δk. It is also worth remarking that the potential (9) is a solution of the equation (10), as can easely be verified.

Few remarks are in order at this point. (a) The disappearance of the metastable region is in agreement with the finding of ref. [7]. The saddle points of the static problem reflect the spontaneous droplet formation dynamics [6]: Suppose that the stable vacuum bubble in a sphere of radius R and created in the false vacuum has the energy $E(R) = 4\pi R^2\sigma - 4\pi R^3\Delta E/3$ where σ and ΔE stand for the surface tension and the (free)energy difference between the two vacua. The critical droplet size, $R_c = \sigma/\Delta E$, beyond which the droplets grow can be identified by the inverse of the highest momentum of the condensate, $R_c^2 = 6/g(\Phi_{vac}^2(0) - \Phi^2)$. This relation establishes a connection between the static and the dynamical properties. (b) The last term in the right hand side of (9) was added by hand to achieve a continuous matching of the partition function at the instability. As a result, (9) reproduces the Maxwell construction for the effective potential $V_{eff}(\Phi) = U_{k=0}(\Phi)$ [8], i.e. it remains unchanged (at the tree level) in the stable region, $\Phi^2 > \Phi_{vac}^2(0)$, and turns out to be constant and continuous for $\Phi^2 \le \Phi_{vac}^2(0)$. (c) The potential (9) contains only one coupling constant, $m^2(k) = \partial_\phi^2 U_k(0) = -k^2$. The corresponding dimensionless coupling constant, $\tilde{m}^2(k) = m^2(k)/k^2 = -1$, is trivially renormalization group invariant. In other words we recover the usual scaling laws in the stable region, while in the unstable region the potential is a fixed point given by (9). (d) Similar tree level potential has been found in the N-component ϕ^4 model with smooth cutoff [9] where the would be unstable and the stable regime join and the naive metastable region is wiped out by the Goldstone modes. While the possibility of having a homogeneous $\psi_k^2(x)$ is an essential part of the argument, our result shows the more general origin of the potential. Furthermore note that the keeping of a single plane wave mode as a saddle point is not justified when a smooth cutoff is used. The analytical structure of the loop corrections in the vicinity of the instability indicates the same potential as well, [10]. The availability of such a diverse derivations and the microscopic dynamics-independent form suggests a more fundamental origin of this potential. We believe that underlying reason of (9) is the Maxwell construction and the actual form can easiest be identified by means of the renormalization group method where the dynamics of each mode can be dealt with individually. In fact, one can show that the differentiability of the renormalization group flow, i.e. the existence of the beta functions, and the nonvanishing of the saddle points as $\Delta k \to 0$ requires the form (9). To see this consider the finite difference

$$\frac{U_{k-\Delta k}(\Phi) - U_k(\Phi)}{\Delta k} = \frac{k^2 + \partial_\Phi^2 U_k(\Phi)}{2L^d\Delta k}\int dx\psi_k^2(x) + \frac{\partial_\Phi^3 U_k(\Phi)}{6L^d\Delta k}\int dx\psi_k^3(x) + \cdots + O(\hbar), \quad (11)$$

where we expanded (7) in the saddle point, $\psi_k(x)$. The convergence of the left hand side requires either the smallness of the saddle point, $\psi_k = O(\Delta k^{1/2})$, or $k^2 + \partial_\Phi^2 U_k(\Phi) = O(\Delta k)$ and $\partial_\Phi^n U_k(\Phi) = O(\Delta k)$. Since the saddle point is nonvanishing the form (9) follows. Note the essential differences between the tree and the loop level renormalization: The former is nonalytic in g and lacks the usual logrithms of the latter. (e) The result (i) can be obtained by assuming (9) in the unstable region and requiring continuity of $\partial_\Phi U_k(\Phi)$ in Φ.

The saddle point approximation includes the softest fluctuations, the zero modes, and one can obtain nontrivial contributions to the correlation functions at $O(\hbar^0)$. In fact, let us insert the product of field variables in the integrand of (2) and follow the successive elimination of the modes in the path integral. The resulting tree level expression for the $2n$-point function on the background field Φ is

$$G^{(0)}_\Phi(p_1,\cdots,p_{2n}) = \frac{\int\mathcal{D}[\hat{k}]\int\mathcal{D}[\alpha]\prod_{j=1}^{2n}\tilde{\psi}_{k_j}(p_j)}{\int\mathcal{D}[\hat{k}]\int\mathcal{D}[\alpha]}$$
$$= \sum_P\prod_{j=1}^{n}\delta(p_{P(2j)}+p_{P(2j-1)})\frac{d(2\pi)^d}{\Omega_d k^d(\Phi)}\rho^2_{p_j} \qquad (12)$$

where $\tilde{\psi}_k(p) = \int d^d e^{-ipx}\psi_k(x)$, Ω_d stands for the solid angle, we integrate over the zero modes characterized by the unit vector $\hat{k}(k)$ corresponding for each value of the cutoff, k, and the phase $\alpha(p)$. The sum in the second equation is over the permutations P of the field variables and $k^2(\Phi) = m^2 - g\Phi^2/6$. Whenevere we have an odd number of fields or $\Phi^2 \geq \Phi^2_{vac}(0)$ or one of the p_j is such that $p_j^2 > m^2 - g\Phi^2/6$, $G^{(0)} = 0$. The integration over the zero modes is reminiscent of the integration over the possible rearrangements of the domain walls in the mixed phase and restores the translation invariance of the correlations. The two point function $G^{(0)}_\Phi(p_1,p_2) = 2\frac{d(2\pi)^d}{\Omega_d k^d(\Phi)}\rho^2_{p_1}\delta(p_1+p_2)$ shows that the Fourier transform of $\tilde{\psi}_p$ can be interpreted as the domain wall structure for a given choice of the zero modes.

The method put forward in this letter, the use of the saddle point approximation for the blocking transformation, can be used to handle some of the systems where large amplitude, inhomogeneous fluctuations are present. We studied here an Euclidean system describing the equilibrium situation. Whenever the time dependence far from equilibrium has nontrivial semiclassical limit this method can be extended to include the real time dependence, [11]. Another natural continuation of this work is the inclusion of the loop corrections in addition to the tree level pieces. The correlation functions computed in such a manner include the fluctuations in a systematic manner which is an improvement compared to ref. [12] where the master equation was used to describe the most probable values of the correlation functions.

FIGURES

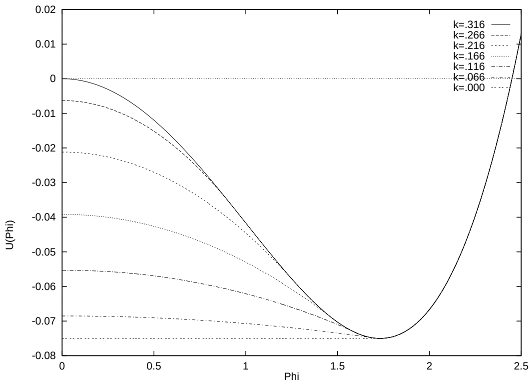

FIG. 1. The potential $U_k(\Phi)$ for different values of k showing the RG evolution towards the Maxwell effective potential.

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